Def • Given two graphs H, G $f: V(H) \to V(G)$ is a homomorphism if $f(e) \in E(G)$ $\forall e \in E(H)$

- $Hom(H,G) = \{$homo. from H to $G\}$
- $hom(H,G) = |Hom(H,G)|$

Q Fix G, which H maximizes $hom(H,G)^{\frac{1}{|V(H)|}}$?

Remark This problem is interesting b/c when $G =$ 

$hom(H,G) = \#$ indep. sets in H $\quad i(H)$

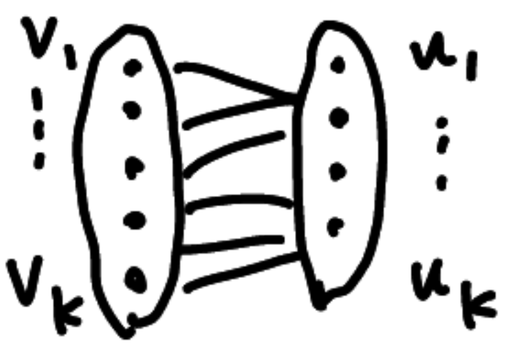
↑ set contains no edge

Thm (Kahn, $G=9$; Galvin-Tetali)

H: n vtx, d-regular bipartite

$$hom(H,G)^{\frac{1}{n}} \le hom(K_{d,d}, G)^{\frac{1}{2d}}$$

Pf $|H|$:



Pick $f \in Hom(H,G)$ u.a.r.

$X_i = f(v_i)$

$Y_i = f(u_i)$

$$\log_2 hom(H,G) = H(X_1, \cdots, X_k, Y_1, \cdots, Y_k)$$

Recall Shearer's ineq.:

$A_1, \cdots, A_r$ covers $[k]$ at least d times.

then $d\,H(X_1, \cdots, X_k) \le \sum H(X_{A_i})$

Take $A_i = \{ j \mid v_j \in N(u_i) \}$ covers $[r]$ d times.

$i = 1, \cdots, k$

$$d\,H(X_1, \cdots, X_k, Y_1, \cdots, Y_k)$$

$$= d\,H(X_1, \cdots, X_k) + d\,H(Y_1, \cdots, Y_k \mid X_1, \cdots, X_k)$$

$$\le \sum H(X_{A_i}) \qquad d \cdot \sum H(Y_i \mid X_1, \cdots, X_k)$$

$$\le d \cdot \sum H(Y_i \mid X_{A_i})$$

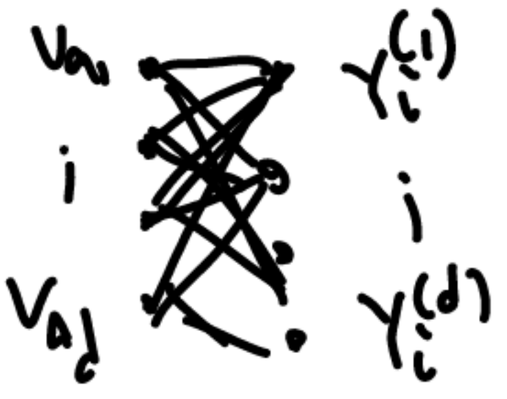
$$\le \sum_{i=1}^{k} H(X_{A_i}) + d\,H(Y_i \mid X_{A_i})$$



$A_i = \{a_1, \cdots, a_d\}$

$$= \sum_{i=1}^{k} H(X_{A_i}, Y_i^{(1)}, \cdots, Y_i^{(d)})$$

where $Y_i^{(1)}, \cdots, Y_i^{(d)}$ are resampled copies of Y_i conditionally independent given X_{A_i}



$$\le k \log_2 \hom(K_{d,d}, G) \qquad \square$$

Generalized Hölder (w/ weights)

Thm $A_1, \cdots, A_m \subseteq [n]$, covers $[n]$ at least k times.

$$f_i : \mathbb{Z}^{A_i} \to \mathbb{R}_{\geq 0}, \quad \pi_{A_i} : \mathbb{Z}^n \to \mathbb{Z}^{A_i} \text{ proj.}$$

$$\sum_{x \in \mathbb{Z}^d} \prod_i f_i(\pi_{A_i}(x)) \leq \prod_i \left(\sum_{y \in \mathbb{Z}^{A_i}} f_i(y)^k \right)^{\frac{1}{k}}$$

pf Sufficient to show for $f_{A_i} : \mathbb{Z}^{A_i} \to \mathbb{Z}$

Sample $(X_1, \cdots, X_n, Y_1, \cdots, Y_m) \in \mathbb{Z}^{n+m}$ s.t.

$0 \leq Y_i \leq f_i(\pi_{A_i}(X_1, \cdots, X_n))$ u.a.r.

$k \log_2 \text{LHS} = kH(X_1, \cdots, X_n, Y_1, \cdots, Y_m)$

$= kH(X_1, \cdots, X_n) + kH(Y_1, \cdots, Y_m \mid X_1, \cdots, X_n)$

$\leq \sum H(X_{A_i}) + \sum k H(Y_i \mid X_{A_i})$

$- \sum_i H(X_{A_i}, Y_i^{(1)}, \cdots, Y_i^{(k)})$

↖ Condition on X_{A_i}. resample Y_i indep.ly

Note that $|\text{supp}(X_{A_i}, Y_i^{(1)}, \cdots, Y_i^{(k)})| \leq \sum_{y \in \mathbb{Z}^{A_i}} f_i(y)^k$

□

Kruskal - Katona

Q Given $\#\Delta$, min #edges?

Complete graph: $\binom{n}{2}$ edges. $\binom{n}{3}$ Δ

Thm Assume $\#\Delta$ in G is $\frac{x^3}{6}$, then $|E| \geq \frac{x^2}{2}$

Pf Sample an ordered Δ (X, Y, Z) u.a.r.

$$2\log_2(6\#\Delta) = 2H(X,Y,Z) \leq H(X,Y) + H(Y,Z) + H(X,Z)$$
$$\leq 3\log_2(2|E|)$$

□

Thm Assume $\#\Delta$ in G is $\binom{x}{3}$, $x \geq 3$, then $|E| \geq \binom{x}{2}$

Pf $H(X,Y,Z) = H(X,Y) + H(Z|X,Y)$
$$= H(Y,Z) + H(X|Y,Z)$$
$$= H(X,Z) + H(Y|X,Z)$$

Claim $2^{H(Y|X,Z)} + 1 \leq 2^{H(Y|Z)}$

$$2^{H(Z|X,Y)} + 2 \leq 2^{H(Z)}$$

Lemma Let Z_1, Z_2 be random variables w/ disjoint supp., $\exists p \in [0,1]$

s.t. $2^{H(Z_1)} + 2^{H(Z_2)} = 2^{H(Z)}$

where $Z = \begin{cases} Z_1 & \text{if } \$ = 0 \\ Z_2 & \$ = 1 \end{cases}$, $\$ \sim \text{bern}(p)$ indep from all rvs.

pf $H(Z) = H(Z, \$) = H(Z|\$) + H(\$)$

$= p\,H(Z|\$=1) + (1-p)\,H(Z|\$=0) + H(\$)$

$= p\,H(Z_1) + (1-p)\,H(Z_2) - p\log_2 p - (1-p)\log_2(1-p)$

$p = \dfrac{2^{H(Z_1)}}{2^{H(Z_1)} + 2^{H(Z_2)}}$: $\log_2\left(2^{H(Z_1)} + 2^{H(Z_2)}\right)$ $\quad$ 0

pf of claim $2^{H(Y,X,Z)} + 2^{H(X,X,Z)} = 2^{H(W,X,Z)}$, $W = \begin{cases} X \\ Y \end{cases}$ if ...

$\Rightarrow 2^{H(Y|X,Z)} + 1 = 2^{H(W|X,Z)}$

$\leq 2^{H(W|Z)} = 2^{H(Y|Z)}$

↑ since $Y|Z = X|Z$

$\$$ indep from Z

$2^{H(Z|X,Y)} + 1 \leq 2^{H(Z|X)}$

$2^{H(Z,X)} + 2^{H(X,X)} = 2^{H(W',X)}$

$\Rightarrow 2^{H(Z|X)} + 1 = 2^{H(W'|X)} \leq 2^{H(W')} = 2^{H(Z)}$

pf of thm Let $H(Z|X,Y) = H(Y|X,Z) = H(X|Y,Z) = \log_2 t$

$$H(Y|Z) \geq \log_2(t+1)$$

$$H(Z) \geq \log_2(t+2)$$

$$\Rightarrow H(X,Y,Z) \geq \log_2 t(t+1)(t+2)$$

$$\Rightarrow x \geq t+2$$

$$\log_2 x(x-1)(x-2) \leq H(X,Y,Z)$$

$$= H(X,Y) + H(Z|X,Y)$$

$$\leq \log_2(2|E|) + \log_2(x-2)$$ □

Remark In a $(d-1)$-uniform hypergraph

$\# K_d^{(d-1)} = \binom{x}{d}$, $x \geq d$, then $\#$ edges $\geq \binom{x}{d-1}$

Sidorenko conjecture

Def $t(H,G) = \dfrac{\hom(H,G)}{|V(G)|^{|V(H)|}}$ homomorphism density

A graph H is **Sidorenko** if

$$t(H,G) \geq t(e,G)^{|E(H)|} \quad \forall\, G$$

Remark • If $G = G(n,p)$, then $\mathbb{E}\, t(H,G) = p^{|E(H)|}$

$$= \left(\mathbb{E}\, t(e,G)\right)^{|E(H)|}$$

Roughly speaking, Sidorenko property says random graph minimizes the density of H

• If H NOT bipartite, then H NOT Sidorenko

pf take $G = e$ □

Thm P_2 :  sido.

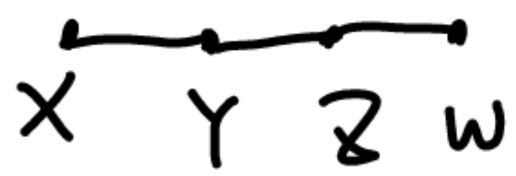
pf $\hom(P_2,G) = \sum_v \deg(v)^2 \leq \dfrac{\left(\sum \deg(v)\right)^2}{n}$, $n = |V(G)|$

$$= \frac{(2m)^2}{n} \qquad m = |E(G)|$$

$$t(K_2,G) \leq \frac{(2m)^2}{n} \Big/ n^3 = \left(\frac{2m}{n^2}\right)^2 = t(e,G)^2$$

Thm P_3 sido

pf Take $XY \in E(G)$ u.a.r.



Condition on Y, resample X indeply. get Z

Condition on Z, resample Y indeply, get W

$$H(X,Y,Z,W) \le \log_2(\hom(P_3,G))$$

$$H(X,Y,Z,W) = \underbrace{H(X,Y,Z)}_{\|} + \underbrace{H(W|X,Y,Z)}_{\|}$$

$$H(X,Y) + H(Z|Y)^{+} \qquad H(W|Z)$$

$$= \underbrace{H(X,Y)+H(Y,Z)+H(Z,W)}_{-H(Y)-H(Z)\ \|} \qquad \underbrace{\qquad\qquad}_{\|}$$

$$\qquad\qquad\qquad

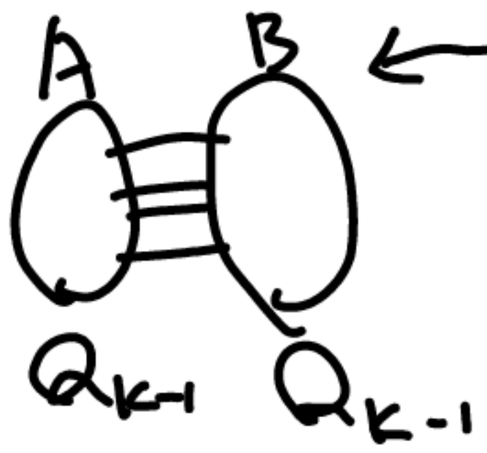
$$H(X,Z) \le H(X) + H(Z) \le 2\log_2 n$$

$$\Rightarrow \frac{\hom(C_4, G)}{n^4} \ge \frac{(2m)^4}{n^8} \quad e$$

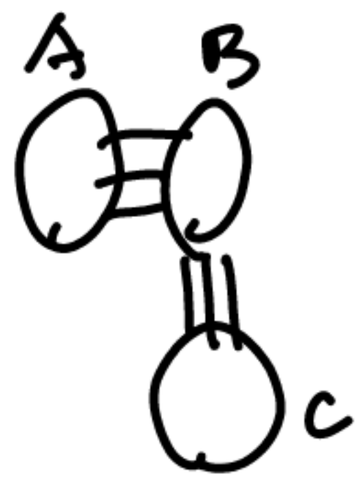
Thm Q_0: •, Q_1: —, Q_2: □, Q_3: 

Q_n sido.

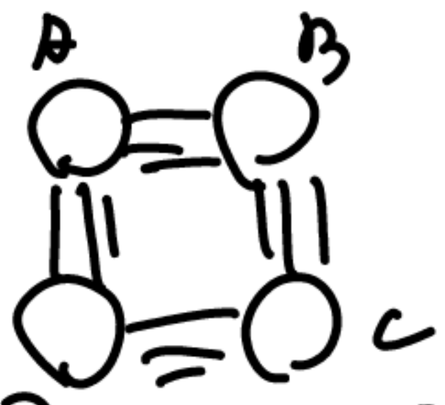
Pf Assume we sampled Q_k. Sample Q_{k+1} by so that the marginal distribution of $A \sim B$



Condition on B resample A indeply



Condition on A, C, resample B indeply



$$\Rightarrow H(Q_{k+1}) = H(A,B,C) + H(D \mid A,C)$$

$$\ge 4H(A,B) - 4H(A) = 4H(Q_k) - 4H(Q_{k-1})$$

Let $D_k = H(Q_k) - 2^k \log_2 n \le \log_2 \hom(Q_k, G)$

$$\Rightarrow D_{k+1} \ge 4D_k - 4D_{k-1}$$

$$D_n \geq 4D_{n-1} - 4D_{n-2}$$

$$4D_{n-1} \geq 16D_{n-2} - 16D_{n-3}$$

$$\vdots$$

$$(k+1)\cdot 2^k D_{n-k} \geq (k+1)2^{k+2} D_{n-k-1} - (k+1)2^{k+2} D_{n-k-2}$$

$$\vdots$$

+)

$$D_n \geq (n-1)2^n D_1 - (n-1)2^n D_0$$

$$= (n-1)2^n \log_2\left(\frac{2n}{n}\right)$$

$D_1 = \log_2\left(\frac{2n}{n}\right)$, $D_0 = 0$ □