Shannon Entropy

Def Let X be a random variable w/ finite support.

$$H(X) := \sum_x -P_X(x) \log_2 P_X(x) \quad \text{, where } P_X(x) = \mathbb{P}(X=x)$$

Remark. If $X \sim \text{unif}\{1, \cdots, k\}$, then $H(X) = \log_2 k$

- $H(X) = \lim_{n\to\infty} \dfrac{\min \#\text{bits encoding } X_1, \cdots, X_n \sim X \text{ iid}}{n}$

Def Let X, Y be two r.v. w/ finite supp.

$$H(X|Y) := H(X,Y) - H(Y) \tag{1}$$

$$= \sum_y \mathbb{P}(Y=y)\, H(X \mid Y=y) \tag{2}$$

$$= \sum_{x,y} -P_{X,Y}(x,y) \log_2 \frac{P_{X,Y}(x,y)}{P_Y(y)} \tag{3}$$

where $P_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$, $P_Y(y) = \mathbb{P}(Y=y)$

pf (1) $\Leftrightarrow$ (3) $H(Y) = \sum_y -P_Y(y) \log_2 P_Y(y)$

$$= \sum_{x,y} -P_{X,Y}(x,y) \log_2 P_Y(y)$$

(3) $\Leftrightarrow$ (2) $\sum_{x,y} -P_{X,Y}(x,y) \log_2 \dfrac{P_{X,Y}(x,y)}{P_Y(y)}$

$$= \sum_{x,y} -P_Y(y) \frac{P_{X,Y}(x,y)}{P_Y(y)} \log_2 \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Prop (Chain rule) $H(X_1, \cdots, X_n) = H(X_1 \mid X_2, \cdots, X_n) + \cdots + H(X_n)$

(uniform bound)

Prop $H(X) \le \log_2 |\mathrm{supp}(X)| = H(\mathrm{unif}\{1,\dots,k\})$

Pf $-x\log_2 x$ is concave, so

$$\frac{\sum -P_X(x)\log_2 P_X(x)}{k} \le -\frac{1}{k}\log_2\frac{1}{k}$$

Jensen

Prop If X, Y are independent, then

$$H(X|Y) = H(X)$$

Pf $H(X|Y) = \mathbb{E}_y H(X|Y=y) = \mathbb{E}_y H(X) = H(X)$

Prop $H(X|Y) \le H(X)$, $H(X|Y,Z) \le H(X|Z)$ ← Ex

Pf

$$\sum_x \sum_y P_Y(y)\left(-\frac{P_{X,Y}(x,y)}{P_Y(y)}\log_2\frac{P_{X,Y}(x,y)}{P_Y(y)}\right)$$

$$\le \sum_x -\left(\sum_y P_{X,Y}(x,y)\right)\log_2\left(\sum_y P_{X,Y}(x,y)\right)$$

Jensen

(subadditivity)

Cor $H(X,Y) \le H(X)+H(Y)$, "=" if and only if X, Y indep ← Ex

(submodularity)

Prop $H(X,Y,Z)+H(X) \le H(X,Y)+H(X,Z)$

Pf $\Leftrightarrow H(Y,Z|X) \le H(Y|X)+H(Z|X)$

$$\mathbb{E}_x H(Y,Z|X=x) \le \mathbb{E}_x H(Y|X=x) + H(Z|X=x)$$

Prop $H(f(X)|X) = 0$, $H(X,f(Y)|Y) = H(X|Y)$

Pf $\mathbb{E}_x H(f(x)|X=x) = 0$ $\quad H(X|Y)+H(f(Y)|X,Y)^{0}$

Prop $H(X|Y) \le H(X|f(Y))$

Pf $H(X|Y) = H(X, f(Y)|Y) = \underbrace{H(f(Y)|Y)}_{=0} + \underbrace{H(X|Y, f(Y))}_{\sim H(X|f(Y))}$

Mutual information

Def $I(X;Y) = H(X) + H(Y) - H(X,Y)$

Prop (1) $I(X;Y) \ge 0$

(2) $I(X;Y) \ge I(X; f(Y))$ ← Ex

Pf (1) subadditivity

Bit-wise information

Q Let $A_1, \cdots, A_m \subseteq [n]$ s.t. $\#\{j \mid i \in A_j\} = \frac{m}{3}$ $\forall i$.

What's max m?

Construction $\binom{[n]}{\frac{n}{3}} \ge \frac{1}{n+1} 2^{h(\frac{1}{3})n}$, $h(p) = H(\text{Bern}(p))$

$= -p\log_2 p - (1-p)\log_2(1-p)$

Upper Bound Let i be a random variable $\sim$ unif$\{1, \cdots, m\}$

$(X_1, \cdots, X_n) \in \{0,1\}^n$, $X_j = 1$ iff $j \in A_i$ be the indicator vector

$\log_2 m = H(X_1, \cdots, X_n) \le H(X_1) + \cdots + H(X_n)$

$= n\, h(\frac{1}{3})$ ⇒

Shearer's inequality

Thm $H(X,Y,Z) \le H(X,Y) + H(Y,Z) + H(X,Z)$

pf $H(X,Y,Z) = H(X,Y) + \underbrace{H(Z|X,Y)}_{\le H(Z)}$

$= H(Y,Z) + H(X|Y,Z)$

$= H(X,Z) + \underbrace{H(Y|X,Z)}_{\le H(Y|Z)}$

$+)$

$H(X,Y,Z)$

Thm Suppose $A_1, \cdots, A_m \subseteq [n]$ (i.e. $\{1,\cdots,n\}$) covers i $\ge k$ times $\forall i \in [n]$

then $k H(X_1, \cdots, X_n) \le \sum_{i=1}^{m} H(X_{A_i})$

where $H(X_A) := H(X_{a_1}, \cdots, X_{a_t})$ if $A = \{a_1, \cdots, a_t\}$

pf Submodularity $\Rightarrow H(X_{A_i}) + H(X_{A_j}) \geq H(X_{A_i \cup A_j}) + H(X_{A_i \cap A_j})$

Replace A_i, A_j by $A_i \cup A_j$, $A_i \cap A_j$, RHS decreases

the family still covers i $\geq k$ times $\forall i \in [n]$

Keep applying this move until we can't (stop in finite time)

why?

WLOG we get $A_1 \supseteq \cdots \supseteq A_k \supseteq \cdots \supseteq A_m$

Since it covers i $\geq k$ times, $A_1 = \cdots = A_k = [n]$

$\forall i$

$\Rightarrow$ RHS $\geq k H(X_{[n]})$ □

Cor (Loomis-Whitney / generalized Hölder)

Let $S \subseteq \mathbb{Z}^n$ finite, $A_1, \cdots, A_m \subseteq [n]$ covers i $\geq k$ times $\forall i$

$P_{A_i} : \mathbb{Z}^n \to \mathbb{Z}^{A_i}$ projection

then $|S|^k \leq \prod_{i=1}^{m} |P_{A_i}(S)|$

pf Sample $(X_1, \cdots, X_n) \in \mathbb{Z}^n$ uniformly from S

$$k \log_2 |S| = k H(X_1, \cdots, X_n) \leq \sum H(X_{A_i})$$

$$\leq \sum \log_2 |\mathrm{supp}(X_{A_i})|$$

$$= \log_2 \prod |P_{A_i}(S)|$$

Triangle-intersecting graph family

Q $\mathcal{F}$ family of subsets of $[n]$ s.t.

$|A \cap B| \geq 1 \quad \forall A, B \in \mathcal{F}$, max $|\mathcal{F}|$?

Easy!

Q $\mathcal{F}$: family of simple graphs on $V = [n]$ s.t.

$A \cap B$ contains a triangle. $\forall A, B \in \mathcal{F}$

max $|\mathcal{F}|$

Construction

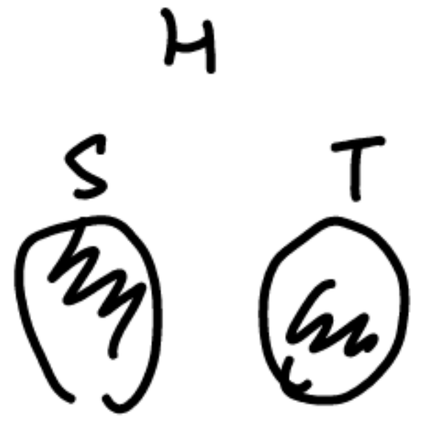
$\#\{$All graphs containing $12, 23, 13\} = \frac{1}{8} 2^{\binom{n}{2}}$

Upper Bound

Thm (Chung, Graham, Frankl, Shearer)

$$|\mathcal{F}| \leq \frac{1}{4} 2^{\binom{n}{2}}$$

Pf (nevan) If $H = K_S \cup K_T$ $S \cup T = [n]$



then $A \cap B \cap H$ has at least one edge.

Let $H_1, \cdots, H_r$ be all such graphs. , $r = \binom{n}{\frac{n}{2}} / 2$

then generalized Hölder $\Rightarrow |\mathcal{F}|^k \leq \prod_{i=1}^{r} |\mathcal{F}|_{H_i}|$

$\mathcal{F}|_{H_i} = \{ A \cap H_i \mid A \in \mathcal{F} \}$, $k = \binom{n-2}{\frac{n}{2}}$

$$|F|_{H_i}| \le 2^{E(H_i)-1} \quad \text{by the previous } \underline{Q}$$

$$= 2^{2\binom{k}{2}-1}$$

$$\Rightarrow |F| \le 2^{\left(2\binom{n}{2}-1\right)\cdot \frac{r}{k}}, \quad \frac{r}{k} = \frac{n(n-1)}{2\frac{n}{2}(\frac{n}{2}-1)} = \frac{(n-1)}{(\frac{n}{2}-1)}$$

$$= 2^{\left(2\binom{n}{2}-1\right)\cdot \frac{n-1}{\frac{n}{2}-1}}$$

$$\le 2^{\binom{n}{2}-2} \qquad \frac{n-1}{\frac{n}{2}-1} \ge 2,$$

<u>Remark</u> Truth: $\frac{1}{8}2^{\binom{n}{2}}$

Rainbow Triangle

G: simple graph w/ R red edges, G green edges, B blue edges

T rainbow triangles

Given R, G, B, max T?

<u>Thm</u> $T^2 \le 8RGB$

<u>Pf</u> Sample $(X,Y,Z) \in V^3$ s.t. XY red YZ green XZ blue
uniformly at random

$$\log_2 \Delta = H(X,Y,Z) \le H(X,Y) + H(Y,Z) + H(X,Z)$$

$$\le \log_2 2R + \log_2 2G + \log_2 2B$$