## Polynomial method

Linear algebraic method: If $n$ linear eq on $m$ var. has only the trivial solution, then $n \geq m$

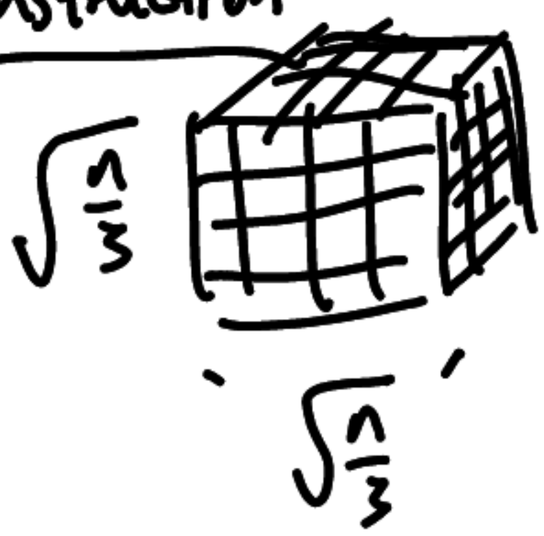
Polynomial method: the vector sp. is a sp of polynomials.

## Joints problem

**Def** Given $n$ lines in $\mathbb{R}^3$. a **joint** is a point in 3 lines w/ linearly indep directions.

**Q** Given $n$ lines, what's the max # joints?

**Construction**




: $n$ lines, $\frac{1}{3\sqrt{3}} \cdot n^{\frac{3}{2}}$ joints

**Thm** (Guth, Katz)

# joints $= O(n^{\frac{3}{2}})$

**Pf** Let $\mathcal{L} = \{\text{lines}\}$, $\mathcal{J} = \{\text{joints}\}$, $J = |\mathcal{J}|$

**Claim** $\exists \ell \in \mathcal{L}$ s.t. $\ell$ contains $\leq 3J^{\frac{1}{3}}$ joints

Pf If not, every line contains $> 3J^{\frac{1}{3}}$ joints.

Take $V = \{ f \in \mathbb{R}[x,y,z] \mid \deg f \le 3J^{\frac{1}{3}} \}$ vector space

Consider equations: $f(p) = 0$ . $p \in J$

$f(x,y,z) = \sum_{a,b,c} \gamma_{a,b,c} x^a y^b z^c$ , then $f(p) = \sum_{a,b,c} \gamma_{a,b,c} p_1^a p_2^b p_3^c = 0$

↑ linear Variables

Claim2 $f \in V$, $f(p) = 0 \ \forall p \in J$

$\Rightarrow f \equiv 0$

Pf If $\exists f \in V$ . $f \neq 0$ , $f(p) = 0 \ \forall p \in J$ , pick $f$ w/ smallest degree

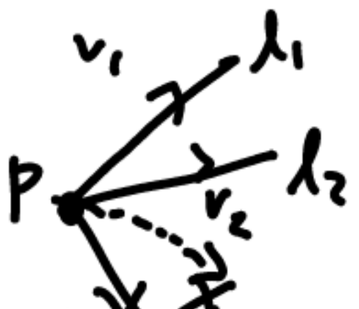
$\forall l \in \mathcal{L}$, write $l = \{at + b \mid t \in \mathbb{R}\}$

Consider $f(at+b) \in \mathbb{R}[t]$ , univariate

$f(at+b)$ has $> 3J^{\frac{1}{3}}$ roots since $l$ contains $> 3J^{\frac{1}{3}}$ points.

FTA $f(at+b) = 0 \ \forall t \in \mathbb{R} \Rightarrow f(p) = 0 \ \forall p \in l, l \in \mathcal{L}$

Consider $\frac{\partial f}{\partial x}(p)$ , $p \in J$ , say $p \in l_1, l_2, l_3$ in $\mathcal{L}$,




$\frac{\partial f}{\partial x}(p) = ? \frac{\partial f}{\partial v_1}(p) + ? \frac{\partial f}{\partial v_2}(p) + ? \frac{\partial f}{\partial v_3}(p)$

$\qquad\qquad\qquad\quad = 0 \qquad\qquad = 0 \qquad\qquad = 0$

since $f(p) = 0 \ \forall p \in l_1$

$= 0$

$\Rightarrow \frac{\partial f}{\partial x} \in V$, $\frac{\partial f}{\partial x}(p) = 0 \ \forall p \in J$ , $\deg \frac{\partial f}{\partial x} < \deg f$ $\quad \lightning$ ? unless $\frac{\partial f}{\partial x} \equiv 0$

Similarly $\frac{\partial f}{\partial y} \equiv 0$, $\frac{\partial f}{\partial z} \equiv 0 \Rightarrow f \equiv 0$ ✱

**Dimension counting** $J$ linear equations on $V$, only the trivial sol.

$$\Rightarrow J \geq \dim V = \#\{x^a y^b z^c \mid a+b+c \leq 3J^{\frac{1}{3}}\}$$
$$a,b,c \geq 0$$
$$= \binom{\lfloor 3J^{\frac{1}{3}} \rfloor + 3}{3} > J \quad ✱$$

□ Claim 1.v

Beginning: $n$ lines $J$ joints, keep removing the line w/ $\leq 3J^{\frac{1}{3}}$ joints until no line. removed $\leq n \cdot 3J^{\frac{1}{3}}$ joints, no joint left

$$\Rightarrow J \leq n \cdot 3J^{\frac{1}{3}} \Rightarrow J \leq 3^{\frac{3}{2}} n^{\frac{3}{2}}$$

□

Remark. The proof works for $\mathbb{R}^d$, where a joint is a point in $d$ linearly indep lines.

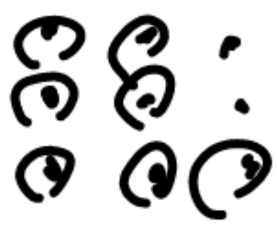
We have $J = O(n^{\frac{d}{d-1}})$.

- If we replace $\mathbb{R}$ w/ $\mathbb{F}$, the proof "almost" works. What's wrong? ex

# Finite field kakeya

**Def** A kakeya set in $\mathbb{R}^2$ is a set that contains a unit segement in each direction.

**Def** A kakeya set in $\mathbb{F}_q^d$ is a set that contains a line in each direction.

**Ex** $\mathbb{F}_3^2$ 

Over $\mathbb{R}$: $\exists$ measure zero construction

Over $\mathbb{F}_q$: Fix $d$, $\exists c_d$ s.t. $\forall$ finite field kakeya set $K$, $|K| \geq c_d q^d$?

**Conj** (Wolff) Yes?

**Thm** (Dvir) Yes! $|K| \geq \binom{q+d-1}{d} \sim \frac{q^d}{d!}$

**pf** Again, let $V = \{ f \in \mathbb{F}_q[x_1, \dots, x_d] \mid \deg f \leq ? \}$ (? $= q-1$)

$f(p) = 0 \quad \forall p \in K$

WTS $f \equiv 0$. If not:

$\forall a \in \mathbb{F}_q^d \setminus \{0\}$, $\exists b \in \mathbb{F}_q^d$ s.t. $at + b \in K \ \forall t \in \mathbb{F}_q^d$

$f(at+b) = 0$ has $q$ roots $\Rightarrow f(at+b) \equiv 0$ as a poly w/ variable $t$.

Write $f(x) = f_0(x) + f_1(x) + \cdots + f_r(x)$, $r = \deg f$, $f_i$: homogeneous deg $i$. $r < q$

$f_0(at+b) + \cdots + f_r(at+b) \equiv 0$

look at $t^r$ part : ↓ $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$

$\rightsquigarrow (a_1 t + b_1)^{\alpha_1} \cdots (a_d t + b_d)^{\alpha_d}$

highest degree part = $a_1^{\alpha_1} \cdots a_d^{\alpha_d} t^{|\alpha|}$

$\Rightarrow f_r(a) = 0$

Thm (Schwartz-Zippel)

Let $f \in \mathbb{F}[x_1, \cdots, x_d]$, $\deg f = n$, $f \not\equiv 0$, $S \subseteq \mathbb{F}$

$\Rightarrow \#\{x \in S^d \mid f(x) = 0\} = |Z(f) \cap S^d| \leq n|S|^{d-1}$

Back to Kakeya, $f_r(a) = 0 \quad \forall a \in \mathbb{F}_q^d \setminus \{0\}$

$f_r(0) = 0$

$|Z(f_r) \cap \pi A_i| = q^d = q \cdot q^{d-1} > r \cdot q^{d-1}$

$\Rightarrow f_r \equiv 0$ ↯

Dim Counting : $|K| \geq \dim V = \binom{q+d-1}{d}$ ∎

pf of S-Z Induct on $d$, $d=1$ : FTA. $n \geq |S|$ trivial. Assume $n < S$

$f(x_1, \cdots, x_d) = \sum_{i=0}^{n} f_i(x_1, \cdots, x_{d-1}) x_d^i$. Say $r$ is the largest s.t. $f_r \not\equiv 0$

$\deg f_r \le n-r \Rightarrow \left| Z(f_r) \cap S^{d-1} \right| \le (n-r)\, S^{d-2}$

For any $(x_1, \dots, x_{d-1}) \in S^{d-1}$, either $f_r(x_1, \dots, x_{d-1}) = 0$

$\Rightarrow f(x_1, \dots, x_{d-1}, t)$ has $\le |S|$ roots w/

or $f(x_1, \dots, x_{d-1}, t)$ has $\le r$ roots $t \in S$

$$\Rightarrow \left| Z(f) \cap S^d \right| \le |S| \cdot (n-r) |S|^{d-2} + r \cdot (|S|^{d-1} - (n-r) |S|^{d-2})$$

$$\le n |S|^{d-1}$$ ∎

Construction (for odd $q$)

Claim $\{(\alpha_1, \dots, \alpha_{d-1}, \beta) \in \mathbb{F}_q^d \mid \alpha_i + \beta^2 \text{ is a square}\}$ $= S$

Contains a line of direction $(\gamma_1, \dots, \gamma_{d-1}, 1) \quad \forall \gamma_1, \dots, \gamma_{d-1}$

If $\frac{\gamma_i^2}{4} + \beta\gamma_i + \beta^2 = \left(\frac{\gamma_i}{2} + \beta\right)^2$

$$\Rightarrow \left(\frac{\gamma_1^2}{4}, \dots, \frac{\gamma_{d-1}^2}{4}, 0\right) + \beta(\gamma_1, \dots, \gamma_{d-1}, 1) \in S$$

Cor $S \cup \{(x_1, \dots, x_{d-1}, 0) \in \mathbb{F}_q^d\}$ is a Kakeya set

$$|S \cup \dots| \le \left(\frac{q+1}{2}\right)^{d-1} \cdot q + q^{d-1} \sim \frac{q^d}{2^{d-1}}$$