1. Entropy (Introduction)

Problem 1.1 (1)**.** Let X, Y, Z be three random variables with finite support. Show that

$$H(X \mid Y, Z) \leq H(X \mid Y).$$

Problem 1.2 (1)**.** Let X, Y be two random variables with finite support. Show that if $H(X, Y) = H(X) + H(Y)$, then X, Y are independent.

Problem 1.3 (1)**.** Let X, Y be two random variables with finite support and let f be a deterministic function. Show that

$$I(X; Y) \geq I(X; f(Y)).$$

Problem 1.4 (2.5, 4)**.** Let $G = (V, E)$ be a simple graph with R red edges, G green edges, B blue edges, and T rainbow triangles. Show that

(1) $T^2 \leq 4RGB$.
(2) $T^2 \leq 2RGB$

Hint for (2): We say that a tuple of three vertices (x, y, z) *forms a rainbow triangle in order* if $\{x, y\}$ is blue, $\{y, z\}$ is red, and $\{x, z\}$ is green. For any vertex x, let $N_g(x)$ and $N_b(x)$ be the set of green neighbours and blue neighbors of x respectively.

In addition, set

$$S_x = \left\{ (x_1, x_2, y, z, z_1, z_2) \in V^6 \middle| \begin{array}{c} (x, x_1, x_2) \text{ and } (z, z_1, z_2) \text{ both form rainbow triangles in order,} \\ \{x, y\} \text{ is blue, and } z \in \{x, y\}, \end{array} \right\},$$

and

$$T_x = \{(a, b_1, b_2, \ell) \in N_g(x) \times N_b(x)^2 \times E \mid \ell \text{ is red}\}.$$

For each $x \in V$, there is an injection from S_x to T_x.

Problem 1.5 (2)**.** The assumption of G being simple in the rainbow triangle problem is important. In a multigraph, we may count the number of rainbow triangle as the number of vertex-triples (v_r, v_g, v_b) such that $v_r v_g$ is blue, $v_g v_b$ is red, $v_r v_b$ is green.

(1) Show that, in this sense, there is a construction with $T^2 = (1 - o(1))8RGB$.
(2) Check that the proof for $T^2 \leq 8RGB$ is still valid.

Thus, the correct constant for this version is 8.

Problem 1.6 (2, 3, 1, 3.5, 5)**.** Let H be a 3-uniform hypergraph with R red edges, G green edges, B blue edges, Y yellow edges and T rainbow $K_4^{(3)}$. Let $c = \frac{T^3}{RGBY}$. Show that

(1) $c \leq 1296$.
(2) $c \leq 54$.
(3) there is a construction with $c = 2$.
(4) there is a construction with $c = 3$.

Can you determine the supremum of c?

Problem 1.7 (4)**.** Let $\mathcal{F}$ be a triangle-intersecting family of graphs on vertex set $[n]$. Show that

$$|\mathcal{F}| \leq 2^{\binom{n}{2}-3}.$$

2. Polynomial Method (Introduction)

Problem 2.1 (2.5)**.** In class, we saw that there is a joints configuration in $\mathbb{R}^3$ with n lines and $\frac{1}{3\sqrt{3}}n^{3/2}$ joints. Can you find a construction with n lines and $(1-o(1))\frac{\sqrt{2}}{3}n^{3/2}$ joints?

Problem 2.2 (1)**.** If we replace $\mathbb{R}$ with $\mathbb{F}_p$, what's wrong with the proof of the joints theorem in class?

Problem 2.3 (2.5, 2.8, 2, 5, 3.5)**.** We say that a set $N \subseteq \mathbb{F}_q^d$ is Nykodim if, for each $p \in \mathbb{F}_q^d$, there is a line ℓ containing p such that $\ell \setminus \{p\} \subseteq N$. For fixed d and large q, show that

(1) $|N| \geq (1-o(1))\frac{q^d}{d!}$
(2) $|N| \geq (1-o(1))\frac{q^d}{2^{d-1}}$
(3) $|N| \geq (1-o(1))q^2$, when $d=2$
(4) $|N| \geq (1-o(1))q^d$

Show that when $d=2$ and $q=p^2$, there is a Nikodym set $N \subseteq \mathbb{F}_q^2$ with $q^2 - cq^{3/2}$ points for some absolute constant $c>0$.

Problem 2.4 (3.6)**.** Let $K \subseteq \mathbb{F}_q^2$ be a Kakeya set. Determine the minimum size of K.

Problem 2.5 (3)**.** Find a construction of a Kakeya set of size $(1-o(1))\frac{q^d}{2^{d-1}}$ when q is even.

Problem 2.6 (2,3)**.** In the basic applications of the polynomial method, we use the property that the space of degree-D polynomials satisfies that vanishing on $D+1$ points on a line implies vanishing on the entire line, and that the space has large dimension. In this problem, we study whether polynomials are the best in this regard.

Let $\mathbb{F}$ be an arbitrary field, D be a positive integer, and V be a vector space of some functions from $\mathbb{F}^d$ to $\mathbb{F}$. Suppose that V satisfies that for any $f \in V$, if it vanishes on $D+1$ points on a line, then it vanishes on every point on that line.

(1) Show that if $|\mathbb{F}| \geq D+1$, then $\dim V \leq (D+1)^d$.
(2) Show that if $|\mathbb{F}|$ is sufficiently large with respect to D and d, then $\dim V \leq \binom{D+d}{d}$.

3. Graph Homomorphism Counting

Problem 3.1 (3.5)**.** Let H be a d-regular graph with n vertices (not necessary bipartite). Show that the number of independent sets

$$i(H) \le i(K_{d,d})^{n/2d} = (2^{d+1}-1)^{n/2d}.$$

Problem 3.2 (4.2)**.** Let H be a graph with n vertices. Show that the number of independent sets

$$i(H) \le \prod_{uv \in E(H)} i(K_{d_u,d_v})^{1/d_u d_v},$$

where d_u denotes the degree of u.

Problem 3.3 (2.3)**.** Prove the Lovász version of the Kruskal–Katona theorem for general uniformity. Let H be a $(d-1)$-uniform hypergraph. Suppose the number of clique of size d, $K_d^{(d-1)}$, in H is $\binom{x}{d}, x \ge d$. Show that the number of edges in H is at least $\binom{x}{d-1}$.

Problem 3.4 (3.5)**.** Prove the q-analog of the Kruskal–Katona problem. Let $\mathcal{A}$ be a family of d-dimensional subspaces of $\mathbb{F}_q^n$ and $\partial\mathcal{A}$ be all the $(d-1)$-dimensional subspaces that are contained in some subspace in $\mathcal{A}$. If $t \ge d$ is a real number that satisfies

$$|\mathcal{A}| = \begin{bmatrix} t \\ d \end{bmatrix}_q \stackrel{\text{def}}{=} \frac{(q^t-1)(q^t-q)\dots(q^t-q^{d-1})}{(q^d-1)(q^d-q)\dots(q^d-q^{d-1})},$$

then

$$|\partial\mathcal{A}| \ge \begin{bmatrix} t \\ d-1 \end{bmatrix}_q = \frac{(q^t-1)(q^t-q)\dots(q^t-q^{d-2})}{(q^{d-1}-1)(q^{d-1}-q)\dots(q^{d-1}-q^{d-2})}.$$

Hint: Sample a set of d linearly independent vectors and try Shearer's inequality. This gives you something close to the desired result.

Problem 3.5 (3)**.** Let $w : \binom{[n]}{2} \to \mathbb{R}_{\ge 0}$ be a weight on the complete graph K_n. Show that

$$\sum_{\{x,y,z\} \in \binom{[n]}{3}} \big(w(xy)w(yz)w(xz)\big)^{1/2} \le \frac{\sqrt{2}}{3}\Big(\sum_{\{x,y\} \in \binom{[n]}{2}} w(xy)\Big)^{3/2}.$$

Note that if we take w to be the indicator function of the edges of a graph, this recovers the Kruskal–Katona theorem without lower order terms.

Problem 3.6 (3)**.** Show the following generalization of the coin-flip lemma we used in class.

Let $Z_1, \dots, Z_r$ be some random variables and $\mathrm{supp}(Z_i)$ be the support of Z_i. Suppose that for any element $x \in \cup_{i=1}^r \mathrm{supp}(Z_i)$, there are at most t many indices i such that $x \in \mathrm{supp}(Z_i)$. It follows that there exists a random variable $\textcircled{\$}$ supported on $\{1, \dots, r\}$, independent from $Z_1, \dots, Z_r$, such that

$$\sum_{i=1}^r 2^{H(Z_i)} \le t 2^{H(Z_{\textcircled{\$}})}.$$

Problem 3.7 (4.9)**.** Let $t \ge d > 0$ be two integers. Let $A_1, \dots, A_{\binom{x}{d}}, x \ge d$ be some sets of size t. Assume $B_1, \dots, B_m$ are sets of size $t-1$ such that for each A_i, there exists at least d distinct $B_j \subset A_i$. Show that $m \ge \binom{x}{d-1}$.

Problem 3.8 (3)**.** Let H be a r-uniform hypergraph and let G be a r-uniform hypergraph with n edges. Show that the number of copies of H in G is at most $O(n^{c(H)})$, where $c(H)$ is the fractional cover number of H. The fractional cover number of H is the minimum of $\sum_{e \in E(H)} w_e$, where $w_e \ge 0$ are real numbers such that $\sum_{e \ni v} w_e \ge 1$ for each vertex $v \in V(H)$.

On the other hand, for large n, construct a graph G such that the number of copies of H in G is $\Omega(n^{c'(H)})$, where $c'(H)$ is the fractional stable number of H. The fractional stable number of H is

the maximum of $\sum_{v \in V(H)} s_v$, where $s_v \geq 0$ are real numbers such that $\sum_{v \in e} s_v \leq 1$ for each edge $e \in e(H)$.

Note that actually we have $c(H) = c'(H)$ by the duality of linear programming.

Problem 3.9 (2.8)**.** We may similarly define the Sidorenko property on 3-uniform hypergraphs. One might expect that any tripartite graph is Sidorenko. However, this is false. Show that loose triangle (see Figure 1) is not Sidorenko.

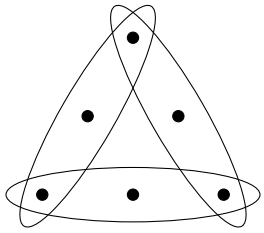


FIGURE 1. 3-uniform loose triangle

Problem 3.10 (3)**.** Let H be a bipartite graph and G be a graph. Show that we have a slightly weaker inequality than the Sidorenko conjecture.

$$t(H, G) \geq t(e, G)^{|E(H)| + \frac{|V(H)|}{2}}.$$

Hint: Find a Sidorenko graph that contains H with only a few more edges than H.

Problem 3.11 (5)**.** Determine whether the Möbius strip $K_{5,5} \setminus C_{10}$ is Sidorenko.

4. Finite Field Kakeya

Problem 4.1 (2.5, 2.5, 5)**.** Let $S \subseteq \mathbb{F}_q^d$. What is the minimum size of S if, for each direction, there is a line ℓ such that

(1) $|S \cap \ell| \geq q - 100$?
(2) $|S \cap \ell| \geq q/2$?
(3) $|S \cap \ell| \geq 100$?

Problem 4.2. Let q be an odd prime power and let

$$L_1 = \{\ell = \{at + b\} | a = (\frac{b_1^2}{4}, \ldots, \frac{b_{d-1}^2}{4}, 0), b = (b_1, \ldots, b_{d-1}, 1), b_1, \ldots, b_{d-1} \in \mathbb{F}_q\},$$

$$K_1 = \cup_{\ell \in L_1} \ell = \{(p_1, \ldots, p_{d-1}, p_d) \in \mathbb{F}_q^d | p_i + p_d^2 \text{ is a square}\},$$

and

$$L_2 = \{\ell = \{at\} | a = (a_1, \ldots, a_{d-1}, 1), a_1, \ldots, a_{d-1} \in \mathbb{F}_q\}$$
$$K_2 = \cup_{\ell \in L_2} \ell.$$

Recall that in class we focused on the vector space

$$V = \text{span}\{x^\alpha | \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d, |\alpha| < (2 - 1/q)qn, \alpha_1 + \cdots + \alpha_{d-1} < qn\},$$

and recall that W_p is the subspace of polynomials satisfies the derivative conditions at point p.

We proved that $\sum_p \text{codim}\, W_p \geq \dim V$ via the polynomial method. Show that, the equality holds up to an $o(1)$ factor when n is large and the Kakeya set is K_1 and K_2 (and the choices of lines are given by L_1 and L_2).

Problem 4.3 (1,3,3,2,3.5)**.** Before Dvir's proof, people did not know how to use the assumption regarding the directions of the lines to its full potential for the finite field Kakeya problem. In this problem, we introduce the *hairbrush* method by Wolff, and show its limit.

(1) In $\mathbb{F}_q^d$, suppose that $\mathcal{L}$ is a collection of lines pointing in each direction. Show that any plane contains at most $q + 1$ of them. We will call this the *Wolff axiom* in this problem, although it is usually more general than just this.
(2) A *hairbrush with stem* ℓ is a subset of $\mathcal{L}$ not containing ℓ such that each of them intersects the line $\ell \in \mathcal{L}$. Show that for any M lines which union to a subset $X \subseteq \mathbb{F}_q^d$, there is one hairbrush with at least $\frac{q^2}{2} M |X|^{-1}$ lines.
(3) In the same setting as the previous problem, show that

$$|X| \geq \frac{1}{3} q^{3/2} M^{1/2}.$$

Hint: for each ℓ' in the hairbrush of ℓ, we can consider the plane spanned by ℓ' and ℓ. There cannot be that many lines in that single plane, so there should be lots of planes spanned by the lines in the hairbrush.

(4) Conclude that for any Kakeya set K in $\mathbb{F}_q^d$, we have

$$|K| \gtrsim q^{\frac{n+2}{2}}.$$

(5) Suppose that $q = p^2$. Consider the points in $\mathbb{F}_q^3$ satisfying

$$x_1^{p+1} + x_2^{p+1} + x_3^{p+1} = 1.$$

Show that there are $\asymp q^{5/2}$ points satisfying this equation. Show that there are $\asymp q^2$ lines in the point set, at most $\lesssim q$ lines of which are in a plane.

5. Joints

Problem 5.1 (5)**.** Recall that $f(r, N, d)$ is the maximum number of sets of size r such that we can choose N sets of size $r-1$ satisfying that each larger set contains at least d of the smaller sets. During the lecture, we see that $f(r, \binom{x}{d-1}, d) \leq \binom{x}{d}$ using the polynomial method.

(1) Show that $f(r, \binom{x}{d-1}, d) \leq \binom{x}{d}$ without using the polynomial method or the joints theorem. Can you show it using the entropy method?
(2) It is known that for any $r \in \mathbb{N}$, if N is sufficiently large, then $f(r, N, 3) = f(3, N, 3)$. Is it also the case that for any $r \geq d$, if N is sufficiently large, then $f(r, N, d) = f(d, N, d)$?

Problem 5.2 (3.5)**.** Recall that $S(\beta_1, \beta_2, \beta_3)$ is the volume of the body

$$\begin{aligned} x_1, x_2, x_3 &\geq 0, \\ x_1 + x_2 + x_3 &\leq 1, \\ x_1 + \beta_1(x_2 + x_3) &\leq \beta_1, \\ x_2 + \beta_2(x_1 + x_3) &\leq \beta_2, \\ x_3 + \beta_3(x_1 + x_2) &\leq \beta_3. \end{aligned}$$

Here, we give two ways of showing $S(\beta_1, \beta_2, \beta_3) \leq S(\overline{\beta}, \overline{\beta}, \overline{\beta})$ where $\overline{\beta} = \frac{1}{3}(\beta_1 + \beta_2 + \beta_3)$.

(1) Method 1: Explicit computation.
   (a) Show the result for the 2-dimensional analogue by explicit computations.
   (b) Show that $S(\beta_1, \beta_2, \beta_3) \leq S(\frac{\beta_1+\beta_2}{2}, \frac{\beta_1+\beta_2}{2}, \beta_3)$.
   (c) Show that $S(\beta_1, \beta_2, \beta_3) \leq S(\overline{\beta}, \overline{\beta}, \overline{\beta})$.
(2) Method 2: Explicit formula.
   (a) Show that $S(\beta_1, \beta_2, \beta_3) = 0$ if $\beta_i \leq 0$ for some $i \in [3]$. Therefore we can assume $\beta_i > 0$ for any $i \in [3]$ from now on.
   (b) Show that if $\beta_1 > 1$, then $S(\beta_1, \beta_2, \beta_3) = S(1, \beta_2, \beta_3)$. Therefore we can assume $\beta_i \in (0, 1)$ for any $i \in [3]$ from now on.
   (c) Let $\gamma_i = (1 - \beta_i)^{-1} - 1 > 0$ for each $i \in [3]$. Show that

$$\begin{aligned} 6S(\beta_1, \beta_2, \beta_3) &= \sum_{\sigma \in S_3} \frac{\gamma_{\sigma(1)}\gamma_{\sigma(2)}\gamma_{\sigma(3)}}{(\gamma_{\sigma(1)} + 1)(\gamma_{\sigma(1)} + \gamma_{\sigma(2)} + 1)(\gamma_{\sigma(1)} + \gamma_{\sigma(2)} + \gamma_{\sigma(3)} + 1)} \\ &= \int_0^1 \prod_{i=1}^{3} (1 - t^{\gamma_i}) \, \mathrm{d}t. \end{aligned}$$

   (d) Let $\overline{\gamma} = (1 - \overline{\beta})^{-1} - 1$, and set

$$f(t) = \frac{1}{3} \sum_{i=1}^{3} t^{\gamma_i} - t^{\overline{\gamma}}$$

   and

$$g(t) = \left(1 - \frac{1}{3} \sum_{i=1}^{3} t^{\gamma_i}\right)^2 + \left(1 - \frac{1}{3} \sum_{i=1}^{3} t^{\gamma_i}\right) \left(1 - t^{\overline{\gamma}}\right) + \left(1 - t^{\overline{\gamma}}\right)^2.$$

   Show that $\int_0^x f(t) \, \mathrm{d}t \geq 0$ for all $x \in [0, 1]$ with equality holding if $x = 1$. Show that $\int_0^1 f(t)g(t) \, \mathrm{d}t \geq 0$ accordingly.
   (e) Show that $S(\beta_1, \beta_2, \beta_3) \leq S(\overline{\beta}, \overline{\beta}, \overline{\beta})$.

Problem 5.3 (1)**.** Show that any critical joints configuration is connected.

Problem 5.4 (1)**.** Verify that σ_p indeed satisfies the conditions of the iterative adjustment lemma.

Problem 5.5 (3.5)**.** Prove the circle packing theorem: for any planar drawing of any finite simple graph G, there exists an assignement of circles $(C_v)_{v\in V}$ to the vertices in V so that the interiors of the circles are disjoint, and that u, v are adjacent to each other in G if and only if C_u and C_v are tangent to each other.

Hint: Prove it for planar drawings with all finite faces being triangles first. Use the iterative adjustment lemma.

Problem 5.6 (4)**.** Prove the multijoints theorem: if there are L_1 red lines, L_2 blue lines, and L_3 green lines, then the number of joints formed by one line of each color is upper bounded by $\sqrt{6L_1L_2L_3}$.

Open problem: what is the optimal constant that we can replace 6 with? We only know that it is between 2 and 6.

Problem 5.7 (4)**.** Prove the joints with multiplicities theorem: for each joint p, let $M(p)$ be the number of triples of lines that form a joint at p; then for any N lines,

$$\sum_p M(p)^{1/2} \lesssim N^{3/2}.$$

Problem 5.8 (3, 2023 Taiwan TST Round 2 Independent Study 2-A)**.** For each positive integer k greater than 1, find the largest real number t such that the following hold: Given n distinct points $a^{(1)} = (a_1^{(1)}, \ldots, a_k^{(1)})$, …, $a^{(n)} = (a_1^{(n)}, \ldots, a_k^{(n)})$ in $\mathbb{R}^k$, we define the score of the tuple $a^{(i)}$ as

$$\prod_{j=1}^{k} \#\{1 \le i' \le n \text{ such that } \pi_j(a^{(i')}) = \pi_j(a^{(i)})\}$$

where $\#S$ is the number of elements in set S, and π_j is the projection $\mathbb{R}^k \to \mathbb{R}^{k-1}$ omitting the j-th coordinate. Then the t-th power mean of the scores of all $a^{(i)}$'s is at most n.

Hint: Using the previous problem, this problem is easy if we replace n with Cn for some large C (say $C = 6$). How to make $C = 1$?

Problem 5.9 (4.5)**.** In $\mathbb{R}^6$, a *joint of* 2*-flats* is a point lying on three 2-dimensional affine subspaces (i.e. 2-flats) such that they are not contained in any hyperplanes. Show that N 2-flats can form at most $O(N^{3/2})$ joints of 2-flats.

Problem 5.10 (2)**.** Show that if $\binom{x}{2}$ lines form $\binom{x}{3}$ joints, then x is an integer, and each line contains the same number of joints.

Problem 5.11 (4.3)**.** Show that if $\binom{x}{2}$ lines form $\binom{x}{3}$ joints, then for any two joints p, p' on ℓ_1 formed by ℓ_1, ℓ_2, ℓ_3 and ℓ_1, ℓ_2', ℓ_3', respectively, we may permute the labels of the lines so that ℓ_1, ℓ_i, ℓ_i' are coplanar for $i = 2, 3$. Conclude that the joints configuration is generically induced.

Hint: If the conclusion is violated, can you put fewer conditions?

Problem 5.12 (3.5)**.** Construct a joints configuration with N lines and $\Omega(N^{3/2})$ joints in $\mathbb{R}^3$ so that any plane contains at most 100 lines.

6. Polynomial Freiman–Ruzsa over $\mathbb{F}_2$

In all problems, we assume that $(G, +)$ is an abelian group. The first several problems guide you through the proof that if $K_A < 1.5$, then A is contained in a small coset.

Problem 6.1 (4)**.** For any $S \subseteq G$, let $\mathrm{stab}(S)$ be the subgroup of elements d with $S + d = S$. Show that if $S = S_1 \cup S_2 \subseteq G$ is a union of two finite subsets, then

$$|S| + |\mathrm{stab}(S)| \geq \min_{i\in[2]} |S_i| + |\mathrm{stab}(S_i)| .$$

Below is an outline that helps you prove this.

(1) Reduce to the case $\mathrm{stab}(S_1) \cap \mathrm{stab}(S_2) = \{0\}$.
(2) Let $H = \mathrm{stab}(S_1) + \mathrm{stab}(S_2)$. Show that if there is $x \in G$ such that $(x + H) \cap S_i \neq 0, S_i$ for all $i \in [2]$, then
$$|S| + 1 \geq \min_{i\in[2]} |S_i| + |\mathrm{stab}(S_i)| .$$
(3) Show that if there exist $x_1, x_2 \in G$ such that $(x_i + H) \cap S_i = \emptyset$ and $(x_i + H) \cap S_j \neq \emptyset$ for all $\{i, j\} = \{1, 2\}$, then
$$|S| \geq \min_{i\in[2]} |S_i| + |\mathrm{stab}(S_i)| .$$
(4) Show that if we cannot find x as in (2), or x_1, x_2 as in (3), then S is a union of some cosets of $\mathrm{stab}(S_1)$ or a union of some cosets of $\mathrm{stab}(S_2)$.

Problem 6.2 (4)**.** Let $A, B \subseteq G$ be two finite subsets. Let $H \subseteq G$ be the subgroup of elements h with $A+B+h = A+B$. Show that if $|A + B| \leq |A|+|B|-1$, then $|A + B| = |A + H|+|B + H|-|H|$. This is a theorem by Kneser.

Below is an outline to help you prove this.

(1) For every $b \in B$, pick A_b, B_b such that
$$b \in B,\ A_b \supseteq A,\ A_b + B_b \subseteq A + B,\ |A_b| + |B_b| = |A| + |B| .$$
Note that we can always pick $(A_b, B_b) = (A, B)$. If there are multiple choices, pick it so that $|A_b|$ is the largest possible. Show that in this case, $A_b + B_b - b = A_b$ for all $b \in B$.
(2) Show that
$$\bigcup_{b\in B} (A_b + B_b) = A + B$$
and
$$\mathrm{stab}(A_b) \subseteq \mathrm{stab}(A_b + B_b)$$
for all $b \in B$.
(3) Apply Problem 6.1 to show that $|A + B| + |H| \geq |A| + |B|$.
(4) Prove Kneser's theorem.

Problem 6.3 (2.5)**.** Assuming Problem 6.2, show that for any finite subset A of G, the following are equivalent.

(1) $K_A < 1.5$;
(2) A is contained in a coset of some subgroup H with $|H| < 1.5\,|A|$;
(3) $A + A$ is a coset of some subgroup of G with $|A + A| < 1.5\,|A|$.

Problem 6.4 (1)**.** Construct some G and finite $A \subseteq G$ so that $K_A = 1.5$ and $A + A$ is not a coset of any subgroup of G.

Problem 6.5 (2)**.** Show that if X, Y are G-valued random variables and $d[X; Y] = 0$, then X, Y are uniform distributions on some cosets of the same subgroup H of G.

Problem 6.6 (4.2)**.** The entropic Balog–Szemerédi–Gowers lemma is closely related to the graph Balog–Szemerédi–Gowers theorem. The statement is as follows. Let $A, B \subseteq G$ be finite subsets of size n, and let $\tilde{G}$ be a bipartite graph on $A \cup B$. Let $K \geq 2$ be a real number. Define $A +_{\tilde{G}} B$ to be $\{a+b : \{a,b\} \in \tilde{E}\}$ where $\tilde{E}$ is the edge set of $\tilde{G}$. If $\left|A +_{\tilde{G}} B\right| \leq Kn$ and $\left|\tilde{E}\right| \geq n^2/K$, then there exists $A' \subseteq A, B' \subseteq B$ both of size at least $K^{-O(1)}n$ such that $|A' + B'| \leq K^{O(1)}n$.

(1) Choose $v \in B$ uniformly at random, and let $U \subseteq A$ be the neighborhood of v in $\tilde{G}$. Show that with positive probability, we have $|U| \geq K^{-O(1)}|A|$, and moreover, the fraction of pairs $\{u_1, u_2\} \in \binom{U}{2}$ with at most $K^{-O(1)}|B|$ common neighbors is at most $K^{-\Omega(1)}$. This is usually called the *path of length* 2 lemma.
(2) Using the path of length 2 lemma, show the path of length 3 lemma: one can choose $A' \subseteq A$, $B' \subseteq B$ both of size at least $K^{-O(1)}n$ so that any $(a,b) \in A' \times B'$ is joined by at least $K^{-O(1)}n^2$ paths of length 3 in $\tilde{G}$.
(3) Show that if for every $s \in A +_{\tilde{G}} B$ there are at least $K^{-O(1)}n$ pairs $\{a,b\} \in \tilde{E}$ with $a+b=s$, then the A' and B' chosen in the path of length 3 lemma also satisfy that $|A' + B'| \leq K^{O(1)}n$.
(4) Show the full graph BSG theorem.
(5) Show the BSG theorem: for every $A \subseteq G$ of size n, if
$$\#\left\{(a,b,c,d) \in A^4 : a+b=c+d\right\} \geq K^{-1}n^3,$$
then there exists $A' \subseteq A$ with $|A'| \geq K^{-O(1)}n$ and $K_{A'} \leq K^{O(1)}$.
(6) Think about how all those are related to the entropic Balog–Szemerédi–Gowers lemma we proved. Can you extract the proof here and make it entropic?

Problem 6.7 (4.8)**.** Prove the polynomial Freiman–Ruzsa conjecture for all abelian groups G with a fixed torsion m. How would you modify the endgame step?

Problem 6.8 (5)**.** Find the smallest $C > 0$ such that the following statement holds true: for all $G = \mathbb{F}_2^n$-valued random variables X_1 and X_2, there exists a subgroup H of G such that
$$d[X_1; U_H] + d[X_2; U_H] \leq C d[X_1; X_2].$$
We gave a proof with $C = 14$. The (current) proof of Gowers–Green–Manners–Tao gives $C = 11$. Jyun–Jie Liao improved it to $C = 10$. On the other hand, our understanding of constructions is relatively poor. A construction with $X_1 = X_2$ and $C \geq 1.5$ would already be interesting (or does it exist?).

Problem 6.9 (?)**.** This is not a precise mathematical problem, but rather a philosophical one. The entropic formulation of the PFR conjecture is equivalent to Marton's conjecture, though the constant C might change throughout the equivalence. This makes the previous problem sensitive to which equivalent formulation we are working in: it is possible that the same problem is easier for one formulation but much harder for the other one. What is the correct (i.e. the easiest) formulation of the PFR conjecture over $\mathbb{F}_2$ so that we can determine the optimal C?

Problem 6.10 (2.5)**.** Prove the "additive Shearer's inequality": if X, Y, Z are independent G-valued random variables, then
$$H(X+Y) + H(X+Z) + H(Y+Z) \geq 2H(X+Y+Z).$$

Problem 6.11 (4)**.** If X, Y are independent G-valued random variables, show that
$$d[X; -Y] \leq 3d[X; Y].$$

7. Incidence Geometry via the Polynomial Method

Problem 7.1 (1)**.** In fact, there is an easier bound than the Kővari–Sós–Turán bound for the point-line incidence problem over arbitrary field. Show that if there are m lines and n points, then the number of incidences is at most $m+n^2$. Note that this is tight up to a constant when $m = \Theta(n^2)$. Similarly, we can bound the number of incidences by $n + m^2$.

Bonus problem: whenever we apply Kővari–Sós–Turán in the proof of Szemerédi–Trotter, we could have actually applied the much easier bound here. We will end up with the same bound. Try to go through the computation yourselves.

Problem 7.2 (2)**.** Show that if $G = (A \sqcup B, E)$ is a bipartite graph with no $K_{s,t}$, then the number of edges can be upper bounded by $s^{1/t} |A|^{1-1/t} |B| + (t-1)|A|$.

Problem 7.3 (3)**.** A surprising application of the Szemerédi–Trotter theorem is the sum-product theorem. The sum-product problem concerns the minimum possible value of $\max(|A+A|, |A \cdot A|)$ given the size of $A \subseteq \mathbb{R}$. This problem will guide you through Szkeley's approach to the sum-product problem.

(1) Consider the set of points $(A+A) \times (A \cdot A) \subseteq \mathbb{R}^2$ and the set of lines

$$\left\{ \{(x,y) \subseteq \mathbb{R}^2 : y = a_1 x - a_1 a_2\} \mid (a_1, a_2) \in A^2) \right\}.$$

Show that there are at least $|A|^3$ incidences.

(2) Apply Szemerédi–Trotter to show that

$$|A+A|\,|A \cdot A| \gtrsim |A|^{\frac{5}{2}}.$$

Conclude that $\max(|A+A|, |A \cdot A|) \gtrsim |A|^{\frac{5}{4}}$.

Problem 7.4 (2.5)**.** Prove the ham snadwich theorem for $d = 2$. Feel free to assume continuity of any functions that you think should be continuous.

Problem 7.5 (3)**.** Prove that given n points and n unit circles on the plane, there are at most $O(n^{4/3})$ incidences. Using this, show that there are at most $O(n^{4/3})$ pairs of points with distance 1 apart among n points on a plane.

Open problem: Try to improve this bound.

Problem 7.6 (1)**.** Recall that we (almost) proved that for m lines and n points in $\mathbb{R}^3$ with no planes containing more than B lines, there are at most

$$\lesssim n + B^{1/3} m^{1/3} n^{2/3} + m^{3/4} n^{1/2}$$

incidences. The original formulation of Guth–Katz actually required that $B \geq m^{1/2}$. Show that this is not necessary by showing how to remove this requirement without knowing the proof at all.

Problem 7.7 (3)**.** In this problem, an *r-rich point* is a point that lies on at least r lines.

(1) Show that for m lines in $\mathbb{R}^2$, the number of r-rich points they form is at most $O(r^{-1}m + r^{-3}m^2)$ using Szemerédi–Trotter.

(2) Using Guth–Katz, show that for m lines in $\mathbb{R}^3$ with at most $m^{1/2}$ lines in any plane, the number of r-rich points they form is at most $O(r^{-1}m + r^{-2}m^{3/2})$ as long as r is greater than some fixed constant.

(3) In the same setting as the previous part, show that

$$\sum_{r=\Omega(1)} r^2 n_r \lesssim m^{3/2} \log m$$

where n_r is the number of points lying on exactly r lines.

Problem 7.8 (2)**.** To actually prove the $N/\log N$ lower bound on the distinct distances, we need to be able to upper bound the number of 2-rich points as well. However, if we naïvely replace r by 2 in the previous problem, we get a false statement. Construct a counterexample to that statement by considering points and lines lying in the quadric $z = xy$. Quadrics like this are called *reguli*.

Problem 7.9 (3.5)**.** Because of the example in the previous problem, the actual Guth–Katz theorem requires that each quadric surface do not contain more than B lines as well. To apply the incidence theorem on Elekes–Sharir reduction, we still need to make sure that the N^2 lines do not concentrate on planes or quadrics.

(1) Show that each plane contains at most N lines from the N^2 lines $(\ell_{pp'})_{p,p'\in\mathcal{P}}$.
(2) Show that each quadric contains at most $4N$ lines from the N^2 lines $(\ell_{pp'})_{p,p'\in\mathcal{P}}$.

Problem 7.10 (4.5)**.** Let $f \in \mathbb{C}[x, y, z]$ be an irreducible polynomial, and let $V = V(f)$ be the irreducible hypersurface in $\mathbb{C}^3$. A point $p \in V$ is *doubly flecnodal* if there are two lines ℓ_1, ℓ_2 through p so that when $f|_{\ell_i}$ is expanded around p, its degree-0, -1, -2 and -3 terms all vanish. Show that if V is not a plane nor a quadric surface, then there exists a polynomial g with $\deg g \lesssim \deg f$ that is not a multiple of f so that if p is doubly flecnodal, then $g(p) = 0$.

Hint: First show that if such g does not exist, then actually for most points $p \in V$ there are at least two lines through p that are completely in V. In other words, V is *doubly-ruled*.

Problem 7.11 (3.5)**.** The Elekes–Sharir framework actually generalizes to arbitrary dimension: it is the incidence geometry ingredient that is still missing. This was done by Bardwell-Evans and Sheffer. The case for $\mathbb{R}^3$ is relatively easier to understand compared to the other dimensions. This problem will guide you through the Bardwell-Evans–Sheffer reduction.

(1) What should $\dim SE(3)$ be? Convince yourselves that it should be 6.
(2) We view $\mathbb{R}^6$ as $\mathbb{R}^3 \times \mathbb{R}^3$ and write each point x as $(\vec{x}^{(1)}, \vec{x}^{(2)})$. For each $x \in \mathbb{R}^6$, consider the map $\rho_x : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\frac{p-\rho_x(p)}{2} = \vec{x}^{(1)} \times \frac{p+\rho_x(p)}{2} + \vec{x}^{(2)}$ for all $p \in \mathbb{R}^3$. Show that ρ_x is always well-defined. Show that most elements of $SE(3)$ are of the form ρ_x for some $x \in \mathbb{R}^6$, and identify those that are not. Also show that $\rho_x \in SE(3)$ for any $x \in \mathbb{R}^6$.
(3) For two points $p, p' \in \mathbb{R}^3$, consider $F_{pp'} = \{x \in \mathbb{R}^6 : \rho_x(p) = p'\}$. Show that $F_{pp'}$ is a 3-dimensional affine subspace of $\mathbb{R}^6$, i.e. a 3-flat.
(4) Show that $\overline{pq} = \overline{p'q'}$ if and only if $F_{pp'}$ and $F_{qq'}$ intersect at a line.

Therefore, if we understood the line-3-flat incidence problem in $\mathbb{R}^6$ really well, we might be able to solve the distinct distances problem in $\mathbb{R}^3$.

Problem 7.12 (2)**.** Suppose $A_1, \dots, A_n$ and $B_1, \dots, B_n$ are non-negative real numbers satisfying $\sum_{i=1}^n A_i = A$, $\sum_{i=1}^n B_i = B$ and $A_1, \dots, A_n \le M$. Show that for any positive reals α, β with $\alpha + \beta \ge 1$, we have $\sum_{i=1}^n A_i^\alpha B_i^\beta \le M^{\alpha+\beta-1} A^{1-\beta} B^\beta$.

Problem 7.13 (3)**.** In the deduction of Guth–Katz, when we are dealing with the cellular part and the cross terms, we already see that we should pick $D = m^{-\frac{1}{4}} n^{\frac{1}{2}}$ to balance the terms and get $m^{\frac{3}{4}} n^{\frac{1}{2}}$. Therefore, if we believe that the algebraic part does not dominate, then we should expect the bound $O(m^{\frac{3}{4}} n^{\frac{1}{2}} + m + n)$ to hold for point-line incidences in $\mathbb{R}^3$.

By induction, compute what you should expect in $\mathbb{R}^d$ assuming that the algebraic part never dominates. The result should be of the form $O_d(m^{\alpha(d)} n^{\beta(d)} + m + n)$. This is indeed the correct prediction, although the proof is far more involved than Guth–Katz.

In the case when the algebraic part does not dominate, show that the number of r-rich points is at most $O_d(m^{\frac{d}{d-1}})$ as long as $r = \Omega_d(1)$. Is it a coincidence that this is the same bound as in the joints problem?