Recall A Kakeya set $K \subseteq \mathbb{F}_q^d$ contains a line in each direction

Last time $|K| \geq \binom{q+d-1}{d} \sim \frac{q^d}{d!}$

$\exists K$ s.t. $|K| \leq \frac{q^d}{2^{d-1}} + O(q^{d-1})$

Thm (Dvir, Kopparty, Saraf, Sudan)

$$|K| \geq \frac{q^d}{(2-\frac{1}{q})^d} \sim \frac{q^d}{2^d}$$

Thm (Bukh, C.)

$$|K| \geq \frac{q^n}{(2-\frac{1}{q})^n} \sim \frac{q^d}{2^{d-1}}$$

Proof idea More equations! Something like $\frac{\partial f}{\partial x}(a) = 0$, ack

Issue $\frac{d^p f}{dx^p} \equiv 0$ if $q = p^n$

$\alpha = (\alpha_1, \dots, \alpha_d)$

Hasse derivative

Taylor: $f(x+y) = \sum_\alpha \frac{f^{(\alpha)}(y)}{\alpha_1! \alpha_2! \cdots \alpha_d!} x^\alpha$ ← multiple of p if α large

notation $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$

Def Hasse derivative $H^\alpha f$ given by

$$f(x+y) = \sum_\alpha H^\alpha f(y)\, x^\alpha$$

Def (Multiplicity) mult(f,p): largest m s.t.

$$H^\alpha f(p)=0 \quad \forall |\alpha|<m, \quad |\alpha| = \alpha_1+\cdots+\alpha_d$$

Prop • $H^\alpha f + H^\alpha g = H^\alpha(f+g)$, $H^\alpha(cf) = cH^\alpha(f)$ i.e. H^α linear

• If $f=\sum_\beta c_\beta x^\beta$, then $H^\alpha f = \sum_\beta \binom{\beta}{\alpha} c_\beta x^{\beta-\alpha}$

where $\binom{\beta}{\alpha} = \prod_i \binom{\beta_i}{\alpha_i}$

• If A is an invertible affine transform, then

$$\text{mult}(f\circ A, p) = \text{mult}(f, A(p))$$

• $\text{mult}(H^\alpha f, p) \geq \text{mult}(f,p) - |\alpha|$

Lemma (Schwartz-Zippel, multi. ver.)

$f\in \mathbb{F}[x_1,\cdots,x_d]$, $\deg f = n$, $f\neq 0$, $S\subseteq \mathbb{F}$

then $\sum_{p\in S^d} \text{mult}(f,p) \leq n\cdot |S|^{d-1}$

Pf induct on d. $d=1$ FTA: $\text{mult}(f,p)=k$

means $f(x) = H^k f(p)(x-p)^k + H^{k+1} f(p)(x-p)^{k+1}\cdots$

has a factor $(x-p)^k$

Suppose $d>1$. If $(x_d - s) \mid f(x)$ for some $s\in S$

Replace f by $f/(x_d-s)$

LHS and RHS both decreases by $|S|^{d-1}$ ✓

If no such s. $f \not\equiv 0$ on each hyperplane $H_s : x_d = s$

Induct hypothesis: $\sum_{p \in S^d \cap H_s} \text{mult}(f|_{H_s}, p) \le n|S|^{d-2}$

$$\Rightarrow \sum_{(p,s)\in S^d} \text{mult}(f,(p,s)) \le \sum_s \sum_p \text{mult}(f|_{H_s},p)$$

$$\le n|S|^{d-1}$$

pf of $|K| \ge \frac{q^d}{(2-\frac{1}{q})^d}$

Consider $f \in \mathbb{F}_q[x_1,\dots,x_d]$, $\deg f < qn$

Claim If $\text{mult}(f,p) \ge (2-\frac{1}{q})n$, then $f \equiv 0$

pf AFSOC $\text{mult}(H^\alpha f, p) \ge (2-\frac{1}{q})n - |\alpha|$

If $\ell = \{at+b \mid t \in \mathbb{F}_q\} \subseteq K$, then

$$\sum_{p\in\ell} \text{mult}(H^\alpha f, p) \ge (2q-1)n - q|\alpha|$$

$$\deg H^\alpha f < qn - |\alpha| \quad \leftarrow \text{If } |\alpha| \le n$$

ν

$$\Rightarrow H^\alpha f|_\ell \equiv 0 \quad \text{if } |\alpha| \in n$$

Write $f = f_0 + f_1 + \dots + f_r$, $r = \deg f$, f_i homogeneous deg i

$r < qn$

$H^\alpha f = H^\alpha f_0 + \cdots + \underline{H^\alpha f_r}$ ← = homogeneous degree $r-|\alpha|$ part of $H^\alpha f$

$H^\alpha f|_\ell = H^\alpha f(at+b)$ degree $\le r-|\alpha|$

$0 = [t^{r-|\alpha|}]\, H^\alpha f(at+b) = [t^{r-|\alpha|}]\, H^\alpha f_r(at+b)$

$= H^\alpha f_r(a)$

$\Rightarrow \mathrm{mult}(f_r, a) \ge n$

$\widetilde{f_r} = f_r(x_1, \cdots, x_{d-1}, 1)$ has mult $\ge n$ at

$\deg = r < qn$ $\quad (x_1, \cdots, x_{d-1}) = (a_1, \cdots, a_{d-1}) \in \mathbb{F}_q^{d-1}$

$\sum_{a \in \mathbb{F}_q^{d-1}} \mathrm{mult}(\widetilde{f_r}, a) \ge q^{d-1} n > q^{d-2} r$ ⇒ w/s $\le$. D

Dim Counting

$\dim \{ f \in \mathbb{F}_q[x_1, \cdots, x_d] \mid \deg f < q^n \} = \binom{q^n+d-1}{d}$

$\mathrm{mult}(f, p) \ge (2-\frac{1}{q})n \Leftrightarrow H^\alpha f(p) = 0 \quad \forall\, |\alpha| < (2-\frac{1}{q})n$

$\binom{(2-\frac{1}{q})n+d-1}{d}$ eq.s

$\Rightarrow K \cdot \binom{(2-\frac{1}{q})n+d-1}{d} \ge \binom{q^n+d-1}{d}$

$K \ge \dfrac{\binom{q^n+d-1}{d}}{\binom{(2-\frac{1}{q})n+d-1}{d}} = (1+o(1)) \dfrac{(q^n)^d}{((2-\frac{1}{q})n)^d} \to \dfrac{q^d}{(2-\frac{1}{q})^d}$ as $n \to \infty$

Pf of $|K| \geq \frac{q^d}{(2-\frac{1}{q})^{d-1}}$

Idea: We used FTA and SZ, try to apply w/ different deg

FTA: $\deg (1-\frac{1}{q})n$; SZ: $\deg n$

$V = \text{Span}\{ x^\alpha \mid |\alpha| < (2-\frac{1}{q})\cdot nq,\ \alpha_1 + \cdots + \alpha_{d-1} < nq \}$

For each $a = (a_1, \cdots, a_{d-1}, 1)$, fix $l_a = \{at+b\} \subseteq K$

$L = \{ l_a \mid a = (a_1, \cdots, a_{d-1}, 1) \in \mathbb{F}_q^d \}$

Def f vanishes to order (n,m) at p along l, $p \in l \in L$

if $\text{mult}(H^{(\alpha,0)} f|_l, p) \geq m - \frac{|\alpha|}{q}$ $\forall |\alpha| < n$

Claim If $f \in V$ vanishes to order $(n, (2-\frac{1}{q})n)$ at p along l

$\forall\, p \in l \in L$, then $f \equiv 0$

Pf By def, $\sum_{p \in l} \text{mult}(H^{(\alpha,0)} f|_l, p) \geq q\cdot(2-\frac{1}{q})n - |\alpha|$

$> \deg\left(H^{(\alpha,0)} f|_l\right)$, $\forall |\alpha| < n$

$\Rightarrow H^{(\alpha,0)} f|_l \equiv 0 \quad \forall l \in L$

Write $f = f_0 t \cdots t f_r$, $r = \deg f$

$[t^{r-|\alpha|}] H^{(\alpha,0)} f(at+b) = H^{(\alpha,0)} f_r(a) = H^{(\alpha,0)} f_r(a_1, \cdots, a_{d-1}, 1)$

$= 0$ $\forall |\alpha| < n$

$$\widetilde{f}_r(x_1,\cdots,x_{d-1}) = f_r(x_1,\cdots,x_{d-1},1)$$

then $\sum mult(\widetilde{f}_r, (a_1,\cdots,a_{d-1},1)) \geq n q^{d-1} > \deg \widetilde{f}_r - q^{d-2}$ $\lightning$

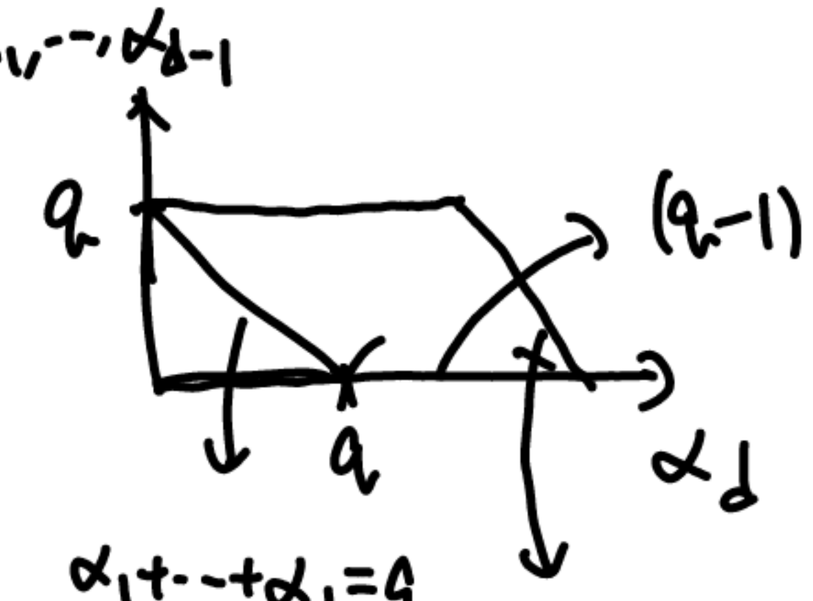
Note that $\deg \widetilde{f}_r < nq$

Dim counting

$$\dim V = (1+o(1)) n^d \cdot Vol(S)$$

$$S = \{ \alpha \in \mathbb{R}^d_{\geq 0} \mid |\alpha| \leq (2-\tfrac{1}{q})\cdot q,\ \alpha_1+\cdots+\alpha_{d-1} \leq q \}$$

S:



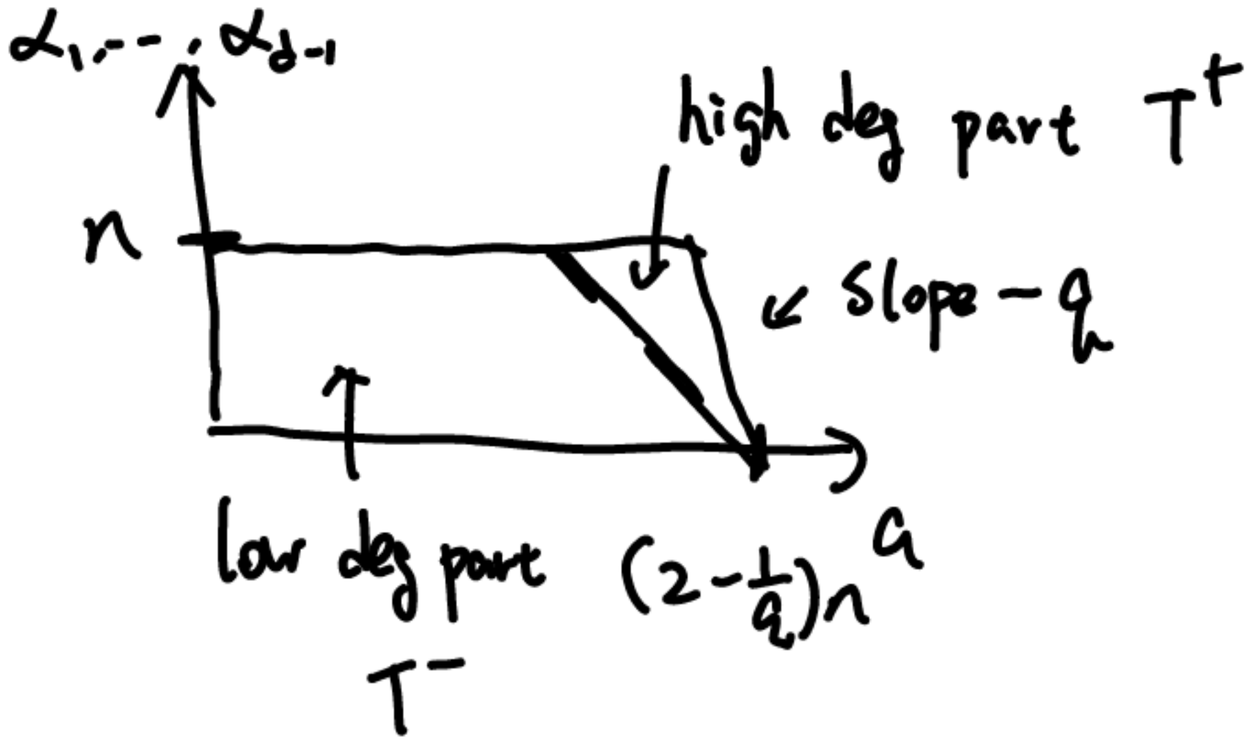
$$Vol(p) = \frac{q^d}{d!} + \frac{q^{d-1}}{(d-1)!}\cdot(q-1)$$

eqs at p: Assume ℓ has direction $(0,\cdots,0,1)$

$$mult(H^{(\alpha,0)} f|_{\ell}, p) \geq (2-\tfrac{1}{q})n - \frac{|\alpha|}{q} \quad \forall |\alpha| < n$$

$$\Leftrightarrow \quad H^{(\alpha,a)} f(p) = 0 \quad \forall a < (2-\tfrac{1}{q})n - \frac{|\alpha|}{q},\ |\alpha| < n$$

$$T = \left\{ (\alpha_1,\cdots,\alpha_{d-1},a) \in \mathbb{Z}^d_{\geq 0} \;\middle|\; \begin{array}{l} \alpha_1+\cdots+\alpha_{d-1} < n \\ a < (2-\frac{1}{q})\lambda - \frac{\alpha_1+\cdots+\alpha_{d-1}}{q} \end{array} \right\}$$



However, each $\ell \ni p$, we see $f(p)=0$, etc.

$\Rightarrow$ Bound low degree part differently

For general ℓ, we have

$$\text{mult}\left(H^{(\alpha,a)} f|_\ell, p\right) \geq \left(2-\tfrac{1}{q}\right)n - \frac{|\alpha|}{q} \quad \forall\, |\alpha| < n$$

$\Leftrightarrow$ $H^{(\alpha,a)}(f \circ A_\ell)(p) = 0 \quad \forall\, (\alpha,a) \in T$, where A_ℓ is a linear trans fixing the first $d-1$ coordinates and $A_\ell(\ell)$ has direction $(0, \sim, 0, 1)$

Note that $H^{(\alpha,a)}(f \circ A_\ell)(p)$ is a linear combination of $H^\beta f(p)$, $|\beta| = |\alpha| + a$

Thus, $H^\beta f(p) = 0 \quad \forall\, |\beta| < \left(2-\tfrac{1}{q}\right)n \Rightarrow H^{(\alpha,a)}(f \circ A_\ell)(p) = 0 \quad \forall (\alpha,a) \in T^-$

$$W_p = \{ f \in V \mid H^{(\alpha,a)}(f \circ A_\ell)(p) = 0 \ \ \forall\, (\alpha,a) \in T,\ \ell \ni p \}$$

eq.s at p = codim W_p

$$W_p = \{ f \in V \mid H^{(\alpha,a)}(f \circ A_l)(p) = 0 \ \forall \ (\alpha,a) \in T^- \ \forall \ l \supset p \} \subsetneq W_p^-$$

$$\cap \{ \text{——} \ , \ \text{——} \ \forall \ (\alpha,a) \in T^+ \ \forall \ l \supset p \} \subsetneq W_p^+$$

$$\operatorname{Codim} W_p^- \le (1+o(1)) \frac{(2-\frac{1}{q})^d n^d}{d!}$$

$$\operatorname{Codim} W_p^+ \le \#\{l \supset p\} \cdot |\bar{T}^+| \qquad \bar{T}^+:$$



$$= \#\{l \supset p\}(1+o(1)) \left(\frac{n^{d-1} \cdot n(1-\frac{1}{q})}{(d-1)!} - \frac{n^{d-1} n(1-\frac{1}{q})}{d!} \right)$$

Cylinder, $\rightarrow$ base = simplex

$$= (1+o(1)) \#\{l \supset p\} \quad \frac{(1-\frac{1}{q}) n^d}{d!} \cdot (d-1)$$

$$\Rightarrow \operatorname{Codim} W_p \le \operatorname{Codim} W_p^- + \operatorname{Codim} W_p^+$$

$$\le (1+o(1)) \frac{n^d}{d!} \left((2-\tfrac{1}{q})^d + \#\{l \supset p\} \cdot (d-1)(1-\tfrac{1}{q}) \right)$$

$$\Rightarrow \text{total } \#\text{eq.} \le \sum \operatorname{codim} W_p$$

$$\le (1+o(1)) \frac{n^d}{d!} \cdot \left(|K| (2-\tfrac{1}{q})^d + \sum_p \#\{l \supset p\} (d-1)(1-\tfrac{1}{q}) \right)$$

$$= (1+o(1)) \frac{n^d}{d!} \left(|K| (2-\tfrac{1}{q})^d + q^d (d-1)(1-\tfrac{1}{q}) \right)$$

$$\Rightarrow |K| \cdot (2-\tfrac{1}{q})^d + (d-1)(1-\tfrac{1}{q}) q^d \ge q^d + d(1-\tfrac{1}{q}) q^d$$

$$\Rightarrow |K| (2-\tfrac{1}{q})^d \ge (2-\tfrac{1}{q}) q^d \qquad \square$$